University of California, Berkeley
Physics 110B, Spring 2004 (*Strovink*)

These exercises should help to reinforce your understanding of component notation, repeated indices, and the transformation properties of vectors and tensors in spacetime:

6.
Consider the Levi-Civita density $\epsilon_{ijk} \equiv 1$ ($ijk = $ even permutation of 123); $\equiv -1$ (odd permutation of 123); $\equiv 0$ (otherwise). It is found, for example, in the cross product

$$(\vec{a} \times \vec{b})_i = \epsilon_{ijk} a_j b_k \ .$$

Note that summation over the repeated indices j and k is implied; their domain is $1 \leq j, k \leq 3$.
(a.)
Show that

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \ ,$$

where δ is the Kronecker delta function (whose elements are those of the unit matrix).
(b.)
The determinant of a 3×3 matrix is given by

$$\det A \propto \epsilon_{ijk} A_{il} A_{jm} A_{kn} \epsilon_{lmn} \ .$$

By considering the number of nonzero terms on the RHS, and comparing it to the number of terms you would have expected for a 3×3 determinant, deduce the constant of proportionality. Express it in terms of a factorial.
(c.)
Guessing the explicit constant of proportionality, write a similar equation for the determinant of a 4×4 matrix. How should ϵ_{ijkl} be defined?

7.
Griffiths Problem 12.55. Don't get fooled by the typo – he means "$\partial^\mu \equiv \partial/\partial x_\mu$".

8.
An object a^μ is a (contravariant) four-vector if it transforms (between frames as defined in Short Course in Special Relativity (SCSR) Fig. 2) according to

$$a'^\mu = \Lambda^\mu{}_\nu a^\nu \ ,$$

where Λ is the (symmetric) 4×4 Lorentz transformation matrix. (Conventionally, the first (superscript) index labels the row and the second (subscript) index labels the column, but this makes no difference for a symmetric matrix.) Covariant four-vectors instead transform according to

$$a'_\mu = a_\nu (\Lambda^{-1})^\nu{}_\mu$$

(otherwise the scalar product $a_\mu a^\mu = a'_\mu a'^\mu$ would not remain invariant for different Lorentz frames). Consider now an (arbitrary) four-tensor $H^{\mu\nu}$. In frame $\mathcal{S}$, $H^{\mu\nu}$ contracts with covariant four-vector a_ν to yield contravariant four-vector b^μ, according to

$$b^\mu = H^{\mu\nu} a_\nu \ .$$

In the frame $\mathcal{S}'$, requiring $H^{\mu\nu}$ to satisfy the transformation properties of a four-tensor, we define $H'^{\mu\nu}$ so that

$$b'^\mu = H'^{\mu\nu} a'_\nu \ .$$

Prove that

$$H'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma H^{\rho\sigma} \ .$$

This defines the Lorentz transformation property of a four-tensor.

9. Consider the antisymmetric *electromagnetic field strength* tensor

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu \ ,$$

where both ∂^μ and A^μ are (contravariant) four-vectors. Prove that $F^{\mu\nu}$ is a four-tensor, *i.e.* it transforms according to the result of Problem 8.